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WHEN THE TWO-PERIOD OPTIMAL POLICY IS OPTIMAL OVER AN INFINITE HORIZON: A NOTE

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DRAFT FOR COMMENT

Problems arising in managing renewable resources, particularly for yield, often take the form:

$$\max E\left\{\sum_{t=1}^{T} \alpha^{t-1} P\cdot(x_t - y_t)\right\}$$

$$\text{s.t.} \quad x_{t+1} = s[y_t, D_t]$$

$$0 \leq y_t \leq x_t$$

where x is the state, y is the decision, and the D_i's are independent, identically distributed random variables. The planning horizon T may be either finite or infinite.

If T is finite, Mendelssohn and Sobel (1977) derive the finite dynamic program:

$$\begin{aligned} &f_0(\cdot) \equiv 0 \\ &f_n(x) = \max_{0 \leq y \leq x} \left\{P \cdot (x-y) + \alpha E f_{n-1}(s[y, D])\right\} \end{aligned} \tag{1}$$

and over an infinite horizon:

$$f(x) = \max_{0 \leq y \leq x} \left\{P \cdot (x-y) + \alpha E f(s[y, D])\right\} \tag{2}$$

In a series of recent papers (Mendelssohn 1978a, b, c) I show how to greatly reduce the effort involved in solving (2). In this paper, I show that for a special case, an optimal policy in (1) for n = 2 is optimal in (2).

Consider the following assumption:

(i) $f(\cdot)$ is concave, continuous

(ii) $Es[y, D]$ is unimodal and differentiable with respect to y.

Conditions which are sufficient for (ii) to be valid are given in Mendelssohn and Sobel (1977). Let y_2^* be the solution to the following equation:

$$E\left\{s^{[1]}[y, D]\right\} = \frac{1}{\alpha} \tag{3}$$

Mendelssohn and Sobel (1977) show that both (1) and (2) have a base stock policy as an optimal policy. That is, there is a y^* such that an optimal policy is to choose:

$$\text{minimum } (x, y^*)$$

Clearly y_2^* is the base stock size at $n = 2$. Theorem 1 proves y_2^* is the base stock size in (2) also.

<u>Theorem 1</u>. <u>Assumptions</u> (i)-(ii) <u>imply</u> y_2^* <u>is an optimal base stock size in</u> (2).

<u>Proof</u>. It is straightforward to show that at y^*,

$$E\left\{s[y^*, D]\right\} \geq y^*$$

(see, for example, Mendelssohn and Sobel 1977).

From equation (1), this implies for $w = E\left\{s[y^*, D]\right\}$:

$$f(w) = p \cdot (w-y^*) + \alpha Ef(s[y^*, D])$$

Applying Jensen's inequality yields:

$$p \cdot (w-y^*) + \alpha Ef(s[y^*, D]) \leq p \cdot (w-y^*) + \alpha f(Es[y^*, D])$$

which implies:

$$p \cdot (w-y^*) + \alpha Ef(s[y^*, D]) \leq p \cdot (w-y^*) + \alpha f(w) = p(w-y^*) + \alpha Ef(s[y^*, D]) \quad (4)$$

Equation (4) implies at y^*, $\alpha Ef(s[y^*, D]) = \alpha f(Es[y^*, D])$. At $y = y_2^*$, $p \cdot (w-y) + \alpha f(w)$ achieves a maximum, which implies $p \cdot (w-y) + \alpha Ef(s[y, D])$ also achieves a maximum at y_2^*. Since a base stock policy is optimal, $y^* = y_2^*$ is the base stock size.

□

What is convenient about theorem 1 is that equation (3) can be solved on nothing more than a pocket calculator. It also underlines a very real problem in using expected value as a criterion for optimization. That is, y_2^* is optimal no matter what the variance of D, so long as the expectation on D is the same. This suggests that when going from deterministic to stochastic models, the expectation of the deterministic objective most likely is not the proper objective function for the stochastic model. There are several ways around this problem. The first is to use utility theory or other related methods to determine the decisionmaker's atitude towards risk. The second is to include smoothing costs of the form:

$$\varepsilon\Big((x_{t-1} - y_{t-1}) - (x_t - y_t)\Big) \quad \text{if } (x_{t-1} - y_{t-1}) > (x_t - y_t)$$

$$\gamma\Big((x_t - y_t) - (x_{t-1} - y_{t-1})\Big) \quad \text{if } (x_t - y_t) > (x_{t-1} - y_{t-1})$$

In a future paper, I will show that for $\varepsilon = \gamma$, this is equivalent to weighting the mean return against the variance of the return. By parameterizing on ε (therefore γ), it is then possible to explore the mean-variance tradeoff.

I suspect, but have not been able to prove, that if assumptions (i)-(ii) are valid, then a two-period optimal policy to the smoothing cost problem is a good approximation to a true infinite horizon optimal policy. This will be explored numerically.

References

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